OPTIMAL PROJECTIONS BY MEANS OF CONVEX LINEAR COMBINATION

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Összefoglalás

A cikk egy új módszert mutat be, melynek segítségével kedvezõbb tulajdonságú keverék-vetületeket kaphatunk az eddig ismerteknél. Max Eckert vetületei adták az alapötletet, õ ugyanis úgy készített új vetületeket, hogy két korábbi vetület számtani közepét vette. Az ezzel a módszerrel elõállított vetület kedvezõbb tulajdonságú, mint a kiindulási vetületek. A vizsgálódásaimból kiderült, hogy általában nem a számtani közép a legkedvezõbb választás. A cikkben fõleg az Eckert-féle vetületekre koncentráltam, és azokra a számtani közép helyett tetszõleges konvex kombinációt alkalmaztam. Kiderült, hogy így kedvezõbb torzulású vetületeket kaphatunk.
Az elõzõ esetben a két kiindulási vetület „keverése" minden pontban ugyanolyan arányú volt. Mivel két vetület közül az egyik általában kedvezõbb tulajdonságokat mutat az Egyenlítõ környékén, míg a másik a pólusok környékén, ezért kézenfekvõ az ötlet, hogy a „keverés" aránya a szélesség függvényében változzon. Azaz az Egyenlítõ környékén nagyobb súlyt kapjon az a vetület, ami ebben a régióban kedvezõbb, míg a pólusok környékén inkább a másik érvényesüljön. Ezzel a módszerrel jelentõsen javíthatók a vetületet jellemzõ teljes torzultsági mérõszámok értékei a kiindulási, illetve az eredeti Eckert-vetületekhez képest.

1 The new method

In the following we introduce a new method of producing projections with favorable properties. The basic idea dates back to Eckert. Using two known projections he obtained a new one by taking the arithmetic mean of equations of the two original projections. We can improve this if we replace the arithmetic mean by one of its generalizations, the *convex linear combination*.

1.1 Convex linear combinations

The basic concept used in this paper is the following:
Definiton. Given real numbers $x_1, x_2, ..., x_n$ and nonnegative numbers $p_1, p_2, ..., p_n \in R$ with $\sum_{i=1}^{n} p_i = 1$, the number

$$s = \sum_{i=1}^{n} p_i x_i$$

is called a *convex linear combination* of $x_1, x_2, ..., x_n$.
The word 'convex' is in the expression because this number is always between the minimum and maximum of the given numbers. (If we have vectors $x_i \in R^n$, then s is in the convex hull of these.)

This concept is going to be useful when we substitute equations of projections for the x_i .
Theoratically, it is possible to 'mix' more projections to get a new, better one, but in this paper we are going to discuss only the case of two. So let us take a closer look at convex combinations of two numbers.
Assume that we have real numbers *x* and *y*, and let $0 \le p \le 1$. Then our convex combination takes the form

$$s = px + (1-p)y.$$

If we think of *x* and *y* as points *X* and *Y* of the number line, then the corresponding point *S* has the following property:

$$\frac{|XS|}{|SY|} = \frac{1-p}{p}$$

This means that *S* subdivides the segment *XY* in ratio *(1-p) : p*.
(If *p*=0 then *s*=*y*, and if *p*=1 then *s*=*x*.) So this little trick enables the two factors to have different effect on the output, which is not the case with the simple arithmetic mean.

1.2 Mixing independent of latitude

Improving Eckert's idea we form the convex combination of two given projections. That is, given $0 \le p \le 1$ and equations of projection $x_1(j, l), y_1(j, l)$ and $x_2(j, l), y_2(j, l)$ we take the following new equations:

$$x(j, l) = p \cdot x_1(j, l) + (1-p) \cdot x_2(j, l)$$
$$y(j, l) = p \cdot y_1(j, l) + (1-p) \cdot y_2(j, l).$$

We expect this to have more favorable distortions than the original ones. In the next section we introduce an additional, very natural 'twist' to it.

1.3 Mixing dependent on latitude

Let us assume that we have two projections. Usually, on certain latitudes one will be better whereas elsewhere we would prefer the other. This leads to the idea of varying the weight *p* by latitude. Everywhere, we will try to assign a higher weight to the more favorable projection, which will therefore have a bigger influence on the outcome. Let us illustrate this on an example.

One obtains Eckert's V. projection by taking the arithmetic mean of the equations of the Mercator-Sanson projection and the Plate Carrée. Let the equations of the Mercator-Sanson projection be

$$x_{MS}(j, l), y_{MS}(j, l)$$

and those for the Plate Carrée,

$$x_{PC}(j, l), y_{PC}(j, l).$$

Then the equations of Eckert's V. projection are

$$x_E(j,l)=\frac{1}{2}\cdot x_{PC}(j,l)+\frac{1}{2}\cdot x_{MS}(j,l)$$

$$y_E(j,l)=\frac{1}{2}\cdot y_{PC}(j,l)+\frac{1}{2}\cdot y_{MS}(j,l).$$

As y_{MS} and y_{PC} are the same, any combination will still give the same result. (For the opposite case see section 1.4 below.) So we will concentrate on the *x*-coordinates.
Near the Equator, Plate Carrée is favorable whereas near the poles Mercator-Sanson is. Therefore at *y*=0° (on the Equator) we take

$$x(0°,l)=1\cdot x_{PC}(0°,l)+0\cdot x_{MS}(0°,l)$$

At the North Pole (*y*=90°) we define the new projection as

$$x(90°,l)=0\cdot x_{PC}(90°,l)+1\cdot x_{MS}(90°,l).$$

Of course between the two extremes there has to be a continuous transition. For example midway through, at *y*=45°, we will define

$$x(45°,l)=\frac{1}{2}\cdot x_{PC}(45°,l)+\frac{1}{2}\cdot x_{MS}(45°,l).$$

The simplest way to obtain such a transition is to vary the weight *p* linearly:

$$x(j,l)=\frac{2}{p}j\cdot x_{MS}(j,l)+\left(1-\frac{2}{p}j\right)\cdot x_{PC}(j,l).$$

Here and from now on we use radian notation for the angle j. It is easy to check that for j=0°, 45°, 90° we get the previous equations.
Note that on all latitudes we formed a convex linear combination of the two projections, only the weight varied.[1] More generally we can take any two projections x_1 and x_2, and a function $p:\left[-\frac{p}{2},\frac{p}{2}\right]\to[0,1]$ as *mixing function* and define

$$x(j,l)=p(j)\cdot x_1(j,l)+(1-p(j))\cdot x_2(j,l)$$

We need not assume neither *p*(0)=0 nor $p\left(\frac{p}{2}\right)=1$. But it is convenient to assume that *p* is differentiable. The reason is that this way one automatically gets differentiable curves as images of meridians (provided that the original projections had this property). For example, this is not the case with the Érdi-Krausz projection.

1.4 When the *y*-equations differ

The above property enables our method to correct disadvantageous properties of the Érdi-Krausz projection. We will only sketch the idea here. The main objection against this projection is that along the 60th latitude the images of meridians are broken. The reason

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[1] For more on this example, see section 3.2 below.

why is that the two projections are patched together along this line. To repair this with only a slight modification of projection properties, one can do the following. Choose a *transition zone* near the 60th latitude, say the zone between the 50th and 70th latitudes. Outside this zone we won't change the original projection. Inside, mix the two projections according to the previous section; choose the mixing function so that on the 50th latitude only the Mercator-Sanson projection prevails and on the 70th, the Mollweide projection does. Between these boundary latitudes, define some continuous transition.

There is, however, an additional problem here. Let the equations of the Mercator-Sanson projection be

$$x_{MS}(j, l),\ y_{MS}(j)$$

and those for the Mollweide projection:

$$x_M(j, l),\ y_M(j)\,.$$

For the 60th meridian to have the same length in the two projections, recall that we have to multiply x_M and y_M by certain constants. After this, y_{MS} and y_M are different functions. Thus it does matter what weigths we use to combine them. The simplest case is when we use the same mixing function as for the x-coordinates. That is, the resulting point P^* will have coordinates

$$P^*\big(p(j)\,x_{MS}(j, l) + (1 - p(j))\,x_M(j, l);\ p(j)\,y_{MS}(j, l) + (1 - p(j))\,y_M(j, l)\big).$$

In this case P^* is on the segment connecting the original points P_{MS} and P_M .

One may also use two different combinations for the two coordinates. On figure 1 we illustrated the case when $p_x = \frac{1}{3}$ and $p_y = \frac{3}{4}$.

Figure 1. When the y-equations differ

In general we can write

$$x(j, l) = p_x(j)\cdot x_{MS}(j, l) + (1 - p_x(j))\cdot x_M(j, l)$$
$$y(j, l) = p_y(j)\cdot y_{MS}(j, l) + (1 - p_y(j))\cdot y_M(j, l)$$

Of course the method can be applied to any other pair of projections. One can also make p_x and p_y depend on the longitude λ. If p_x and p_y are independent of λ then the combination of two cylindrical projections is again cylindrical.

2 Applications to azimuthal projections

I will show how the method works for azimuthal projections, because here it is simple to carry out the calculations. To compare projections we will use mean square errors. In this paper we use only the original Airy, the modified Airy, and the Airy-Kavrayskiy criterion for this aim. It is well known that some of the azimuthal projections project only the open

hemisphere. So in our calculations of mean square errors, we take into consideration only the region T between the pole and latitude 5°.
Let us see the formulae. In general, the expression for the mean square error E is

$$E^2 = \frac{1}{m(T)} \int_T \varepsilon \, dT$$

where $m(T)$ denotes the area of T and ε is one of criteria ε_{AO}, ε_{AM} and ε_{AK}. For our region, this takes the form

$$E^2 = \frac{1}{2\pi\left(1-\sin 5°\right)} \cdot \iint_T \varepsilon \, d\varphi \, d\lambda$$

Our projection is azimuthal, so

$$E^2 = \frac{1}{2\pi\left(1-\sin 5°\right)} \cdot 2\pi \cdot \int_{5°}^{90°} \varepsilon \, d\varphi \; = 1.09547712 \int_{5°}^{90°} \varepsilon \, d\varphi$$

In the case of azimuthal projections, mixing means mixing of *functions of radius of parallels*. We will illustrate the method on an example.

2.1 Mixing of Lambert azimuthal equal-area and stereographic projections

Let us consider the Lambert equal-area and the stereographic projections with functions of the radius of parallels

$$q_1(\beta) = 2\sin\tfrac{\beta}{2} \quad \text{and} \quad q_2(\beta) = 2\,tg\,\tfrac{\beta}{2}$$

2.1.1 Constant mixing function

First we consider the case when the mixing function is constant (independent of latitude), that is,

$$p(\varphi) = p$$

Then the function of radius of parallels of the new projection becomes

$$q(\beta) = 2p\sin\tfrac{\beta}{2} + 2(1-p)\,tg\,\tfrac{\beta}{2}$$

At each point the stereographic projection has distortion of area $\tau \geq 1$ and the Lambert projection is equal-area. Thus all their convex combinations also have $\tau \geq 1$. This implies $E_{AM} = E_{AO}$. In the following we use E_{AO} and E_{AK} to compare our projections and to find an optimal p.
The mean square errors of our original two projections are

	E_{AO}	E_{AK}
Lambert equal-area	2.268183	1.276001
Stereographic	5.862292	2.552019

Our calculations showed that p=0.813 minimizes E_{AO} at E_{AO} =1.825473. E_{AK} takes on its minimum value 1.112005 at p=0.8345. In both cases these values are far better than those in the table.

2.1.2 Linear mixing function

We hope for further improvement by using a linear mixing function (see also section 1.3). That is, we assume $p(j) = Aj + B$. As we know that $0 \le p \le 1$, and we are interested only in a hemisphere (that is, $0 \le j \le \frac{p}{2}$), we can substitute $x = \frac{2}{p} j$. This allows us to look for the optimal mixing function in the form

$$p(j) = Ax + B.$$

Note that now we have a function $p:[0,1] \to [0,1]$.

For the optimization we used an approximate method (like in the case of the previous section too). It is necessary that $|A| \le 1$ and $0 \le B \le 1$. Otherwise we can think of A and B as independent variables. Our task is then to minimize a function $R^2 \to R$, with variables A and B, on the rectangle $[-1,1] \times [0,1]$. The values of this function are the mean square errors corresponding to A and B. In the actual approximation we calculated values on a 40×40 grid and fitted contour lines on these data. The optimum turnes out to be

$$p(j) = -0.18x + 1.$$

This gives mean square errors E_{AO} =1.759690 and E_{AK} =1.080955. In both cases these are better than those values obtained by constant mixing function.

3 Eckertian projections with improved features

In his 1906 paper Eckert published six projections. We will focus on two of them, Eckert III and V. These are both arithmetic means of the Plate Carrée and another projection. For Eckert III it is the Apian II, and for Eckert V, the Mercator-Sanson projection. More precisely, he multiplied both equations by a certain constant to make them preserve the area of the Earth. Let us focus first on these 'area constants.'

3.1 The area constants

We inquire into the calculation of these constants.The image of the Earth has two symmetry axes in these projections, the Equator and the Prime Meridian. So it is enough to deal with one quadrant. Before multiplying by the area constant the y-equation is simply

$$y = j \; . \tag{1}$$

Therefore the area of the quadrant is

$$T = \int_0^{\frac{p}{2}} x(j, 180°)\, dj \; .$$

(This formula is valid always when (1) holds.) Taking the radius of the Earth as a unit, one-fourth of the surface area of the Earth is $A = \frac{4p}{4} = p$. The ratio of these two, $\frac{A}{T}$, is exactly the square of the area constant (because one multiplies both x and y by it). Thus

$$c^2 = \frac{p}{\int_0^{\frac{p}{2}} x(j, 180^\circ)\, dj} \quad . \tag{2}$$

3.2 Projections of type Eckert V

3.2.1 Constant mixing function

In this case we have two cylindrical projections (Plate Carrée and Mercator-Sanson), and we mix them with a constant function ($p(j) = p$). The equations of the new projections are the following:

$$x = l \cdot (1 - p + p \cos j)$$

$$y = j \ .$$

These form a family of projections. If p=0 then we get the Plate Carrée, if p=1, the Mercator-Sanson projection; and if $p = \frac{1}{2}$, Eckert's V projection.

The area constant is the following:

$$c^2 = \frac{p}{\int_0^{\frac{p}{2}} (1 - p + p \cos j) p \, dj} = \frac{1}{[(1 - p) j + p \sin j]_0^{\frac{p}{2}}} = \frac{1}{(1 - p)\frac{p}{2} + p} \quad ,$$

that is,

$$c = \frac{1}{\sqrt{(1 - p)\frac{p}{2} + p}} \ .$$

Similar approximate calculations show that the following values of p give the optimal mean square errors in our three cases:

	optimal p	mean square error
$\mathbf{E_{AO}}$	0.425	0.931864
$\mathbf{E_{AM}}$	0.433	0.942006
$\mathbf{E_{AK}}$	0.334	0.284857

All these values are even better than those for the Eckert V projection. Figure 2 shows this for the case of the Airy-Kavrayskiy criterion.

Figure 2. Mean square errors E_{AK} for different values of p

3.2.2 Linear mixing function

Now the mixing function has the following form:

$$p(j) = Ax + B$$

where $x = \frac{2}{p} j$. So the equations of projection are

$$x = \left[\left(A\frac{2}{p}j + B\right)\cos j + 1 - \left(A\frac{2}{p}j + B\right)\right] l$$

$$y = j\ .$$

Following Eckert, we will have to multiply by another area constant. It is of the form

$$c = \sqrt{\frac{p}{T_{5,1}(A,B)}}\ ,$$

where $T_{5,1}$ is a linear function of A and B.[2]
To narrow down the possible mixing functions to a 1-parameter family, we imposed an additional condition on them, namely that their average (integral) be the same as the optimal constant in the previous section. We obtained the following optimums:

	A	B	mean square error
E_{AO}	0	0.425	0.931864
E_{AM}	0	0.433	0.942006
E_{AK}	0.037	0.3155	0.284355

The data show that there is no improvement when we measure errors according to E_{AO} or E_{AM}.

3.2.3 Cubic mixing function

[2] For the actual expression we need the following integral:

$$T_{5,1}(A,B) = p \cdot \int_0^{\frac{p}{2}} \left(A\frac{2}{p}j + B\right)\cos j + 1 - \left(A\frac{2}{p}j + B\right) dj$$

By standard integration methods, one gets

$$T_{5,1}(A,B) = \left(p - \frac{p^2}{4} - 2\right)A + \left(p - \frac{p^2}{2}\right)B + \frac{p^2}{2}\ .$$

We turn now to the case when we mix the two projections by a cubic function. For concreteness, we will consider affine images of the function[3] $3x^2$ - $2x^3$. These are of the form

$$p(j) = 3\,D\,x^2 - 2\,D\,x^3 + E \ .$$

The equations of projection are:

$$x = \left[\left(3\,D\,x^2 - 2\,D\,x^3 + E\right)\cos j + 1 - \left(3\,D\,x^2 - 2\,D\,x^3 + E\right)\right] l$$
$$y = j$$

where $x = \dfrac{2}{p} j$.

The corresponding area constant is

$$c = \sqrt{\frac{p}{T_{5,3}(D,E)}} \ ,$$

where, similarly to the previous section, one has

$$T_{5,3}(D,E) = \left(p - \frac{p^2}{4} + \frac{24}{p} - \frac{96}{p^2}\right) D + \left(1 - \frac{p^2}{2}\right) E + \frac{p^2}{2} \ .$$

Again, we take into account only those functions whose integrals agree with the previously found optimal constants. The results are in the following table.

	D	E	mean square error
$\mathbf{E_{AO}}$	0	0.425	0.931864
$\mathbf{E_{AM}}$	0	0.433	0.942006
$\mathbf{E_{AK}}$	0.0077	0.3301	0.282860

Just like with linear functions, we could improve only E_{AK}.

3.3 Projections of type Eckert III

We turn our attention now to the closely related Eckert III projection. Thus, we will mix the Plate Carrée and Apian's II. projection. The equations of Eckert's III. projection are:

$$x = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4j^2}{p^2}}\right) l$$
$$y = j \ .$$

It has area constant

$$c = \frac{4}{\sqrt{p\,(4+p)}} \ .$$

[3] The function $f(x)=3x^2-2x^3$ has the following favorable properties:
$f(0)=0, f(1)=1, f'(0)=0, f'(1)=0$ and $f:[0,1] \to [0,1]$ is a bijection.

3.3.1 Constant mixi ng function

Mixing the two original projections with a constant function $p(j) = p$ we get the following equations for the new projection:

$$x = \left(p\sqrt{1-\tfrac{4j^2}{p^2}} + 1 - p \right) l$$

$$y = j \quad .$$

These form a family of projections again. If p=0$ then we get the Plate Carrée, if p=1, the Apian II, and if $p = \dfrac{1}{2}$, Eckert's III. projection.

In the case of constant mixing function the area constant is:

$$c = \frac{1}{\sqrt{p\frac{p^2}{8} + (1-p)\frac{p}{2}}} \quad .$$

The calculations show that the following values of p are optimal.

	optimal p	mean square error
$\mathbf{E_{AO}}$	0.614	0.964716
$\mathbf{E_{AM}}$	0.627	0.976775
$\mathbf{E_{AK}}$	0.566	0.284335

All these values are better than those for the Plate Carrée, Apian II, and the Eckert III projection.

3.3.2 Linear mixing function

Now the mixing function is the following:

$$p(j) = Ax + B$$

where $x = \dfrac{2}{p} j$. So our equations are

$$x = \left[\left(A\frac{2}{p} j + B \right) \sqrt{1-\tfrac{4j^2}{p^2}} + 1 - \left(A\frac{2}{p} j + B \right) \right] l$$

$$y = j \quad .$$

Using equation (2) (similarly to the previous section) we get the area constant

$$c = \sqrt{\frac{p}{T_{3,1}(A,B)}} \; ,$$

where

$$T_{3,1}(A,B)=\frac{p^{2}}{2}\left[1-\frac{A}{6}-\left(1-\frac{p}{4}\right)B\right].$$

Here too we look for the optimal function only in a special family of linear functions (as in section 3.2.2). The results are similar to those for projections of type Eckert V. If we use the Airy-Kavrayskiy criterion then the optimal linear mixing function is better than constant functions. The best choise is the function with A=0.052 and B=0.54. In this case E_{AK} =0.283942.

3.3.3 Cubic mixing function

Again, we examine the same cubic mixing functions as in the case of projections of type Eckert V. The general formula of these functions is

$$p(j) = 3D\,x^{2} - 2D\,x^{3} + E \ ,$$

where $x = \frac{2}{p} j$.

In this case the area constant is

$$c = \sqrt{\frac{p}{T_{3,3}(D,E)}} \ ,$$

where

$$T_{3,3}(D,E)=p^{2}\left(\frac{1}{2}-\frac{77}{240}D+\left(\frac{p}{8}-\frac{1}{2}\right)E\right).$$

Similar to previous sections we obtain that only in the case of Airy-Kavrayskiy criterion there is a better cubic function than constant functions. But we find the surprising result that this optimal cubic function is less favorable than the optimal linear mixing function. The optimal function has parameters D=0.0032 and E=0.5644. In this case E_{AK}=0.284325.

Figure 3. Projection of type Eckert III, mixed with the optimal constant mixing function

References

[1] Lev M. Bugayevskiy–John P. Snyder: *Map Projections – A Reference Manual*, Taylor&Francis, London, 1995.

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[3] Karl-Heinz Wagner: *Die unechten Zylinderprojektionen*, Archiv der deutschen Seewarte 51. Bd., Nr. 4., Hamburg,1932.

[4] Max Eckert: *Neue Entwürfe für Erdkarten*, Pettermanns Mitteilungen v. 52, no. 5.